

MAPPING PROPERTIES OF THE HEAT OPERATOR ON EDGE MANIFOLDS

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ABSTRACT. We consider the heat operator on spaces with complete and incomplete edge metrics. We establish its mapping properties, recovering and extending the classical results from smooth manifolds and conical spaces. The estimates yield short-time existence of solutions to certain semilinear parabolic equations. Our discussion reviews and generalizes the earlier work by Jeffres and Loya.

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Edge metrics of complete and incomplete type provide interesting examples of geometries that include asymptotically hyperbolic, asymptotically cylindrical and conic spaces. Incomplete and complete edge metrics behave differently from a geometric perspective. However, these metrics lend themselves to a parallel approach when one wants to do constructions of an analytic flavour. For example, to analyze asymptotics of solutions to the heat equation near spatial infinity in the complete case, one must introduce a geometric compactification. The resulting complete edge metrics, like their incomplete cousins, give rise to differential operators that are degenerate or singular in a sufficiently controlled manner that generalization of classical results is possible. In particular, we consider the Laplacian on manifolds with either type of edge metric and

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establish regularity properties of solutions to the heat equation. We study the cases in parallel, reviewing and extending previous work by Jeffres and Loya for conic and b-metrics [JL03, JL04].

We begin by defining these metrics precisely. Let \overline{M} be an m -dimensional compact manifold with boundary ∂M , where ∂M is the total space of a fibration $\phi : \partial M \rightarrow B$, and the fibre F and base B are closed manifolds. We consider a defining function $x : C^\infty(\overline{M}) \rightarrow \mathbb{R}^+ \cup \{0\}$ of the boundary ∂M with $x^{-1}(0) = \partial M$ and $dx \neq 0$ on ∂M . Using the integral curves of $\text{grad}(x)$ we identify a collar neighbourhood $U \subset \overline{M}$ of the boundary ∂M with $[0, 1) \times \partial M$, where ∂M is identified with $\{0\} \times \partial M$.

Definition 1.1. A Riemannian manifold $(\overline{M} \setminus \partial M, g) := (M, g)$ has a simple non-iterated

- (i) incomplete edge at B if, with respect to the identification $U \setminus \partial M \cong (0, 1) \times \partial M$, the Riemannian metric $g = g_0 + h$ and g_0 attains the form

$$g_0 \upharpoonright U \setminus \partial M = dx^2 + x^2 g^F + \phi^* g^B,$$

- (ii) complete edge at B if, with respect to the identification $U \setminus \partial M \cong (0, 1) \times \partial M$, the Riemannian metric $g = g_0 + h$ and g_0 attains the form

$$g_0 \upharpoonright U \setminus \partial M = \frac{dx^2}{x^2} + \frac{\phi^* g^B}{x^2} + g^F,$$

where g^B is a Riemannian metric on the closed manifold B , g^F is a symmetric 2-tensor on the fibration ∂M restricting to a Riemannian metric on each fibre F , and $|h|_{g_0} = O(x)$ as $x \rightarrow 0$.

We will refer to either of these metrics as an edge metric. As the names imply, incomplete or complete edge metrics are incomplete or complete as Riemannian metrics. In the complete case ∂M is at infinite distance from any point in the interior of \overline{M} ; note that the two types of edge metrics are conformal in the interior of \overline{M} . Examples of complete edge metrics include asymptotically hyperbolic (conformally compact or 0-metrics) and asymptotically cylindrical (b-metrics) metrics, with $\dim F = 0$ and $\dim B = 0$ respectively. The product metric on $\mathbb{H}^{b+1} \times \mathbb{S}^f$ provides an example of a complete edge metric with trivial fibration structure. Examples of incomplete edge metrics include conic metrics and conformal compactifications of asymptotically hyperbolic metrics.

We consider a slightly restricted class of edge metrics, requiring that $\phi : (\partial M, g^F + \phi^* g^B) \rightarrow (B, g^B)$ be a Riemannian submersion. Recall that if $p \in \partial M$, then $T_p \partial M$ splits into vertical and horizontal subspaces as $T_p^V \partial M \oplus T_p^H \partial M$, where $T_p^V \partial M$ is the tangent space to the fibre of ϕ through p and $T_p^H \partial M$ is the orthogonal complement of this subspace. The new condition on g_0 implies that the restriction of the tensor g^F to $T_p^H \partial M$ vanishes. Moreover, in the case of incomplete edge metrics we need to assume that the Laplacians associated to g^F at each $b \in B$ are isospectral. We summarize these additional conditions in the definition below.

Definition 1.2. Let (M, g) be a Riemannian manifold with an edge metric. This metric $g = g_0 + h$ is said to be feasible if

- (i) $\phi : (\partial M, g^F + \phi^* g^B) \rightarrow (B, g^B)$ is a Riemannian submersion;
- (ii) if the edge metric is incomplete, the Laplacians associated to g^F at each $b \in B$ are isospectral.

Our perspective for establishing regularity for solutions to the heat equation is as follows: the mapping properties of the corresponding heat operator are encoded in the asymptotic behaviour of the heat kernel on an appropriate blowup of the heat space, as established by Mazzeo and the third author [MV10] for incomplete and by Albin [ALB07] for complete edge metrics. The regularity is discussed in terms of spaces with bounded edge derivatives and appropriate Hölder spaces, which take into account the underlying singular or asymptotic geometry. To make this precise we introduce the notion of edge vector fields \mathcal{V}_e .

Definition 1.3. Let \overline{M} be a compact manifold with boundary ∂M being the total space of a fibration $\phi : \partial M \rightarrow B$ with fibre F . Define the vector space \mathcal{V}_e to be the space of vector fields smooth in the interior of \overline{M} and tangent at the boundary ∂M to the fibres of the fibration.

The space \mathcal{V}_e is closed under the ordinary Lie bracket of vector fields, hence defines a Lie algebra. In local coordinates \mathcal{V}_e can be described as follows. Let $y = (y_1, \dots, y_b)$, $b = \dim B$ be the local coordinates on B lifted to ∂M and then extended inwards. Let $z = (z_1, \dots, z_f)$, $f = \dim F$ restrict to local coordinates on F along each fibre of ∂M . Then (x, y, z) are the local coordinates on \overline{M} near the boundary and the edge vector fields \mathcal{V}_e are locally generated by

$$\left\{ x \frac{\partial}{\partial x}, x \frac{\partial}{\partial y_1}, \dots, x \frac{\partial}{\partial y_b}, \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_f} \right\}.$$

Definition 1.4. Let (M, g) be a Riemannian manifold with an edge metric.

- (i) Let $C_e^0(M, g)$ denote the space of functions that are continuous up to the boundary of \overline{M} . The space $C_e^k(M, g)$ with $k \in \mathbb{N}$ consists of all $u \in C_e^0(M, g)$ such that for any choice of edge vector fields $V_j \in \mathcal{V}_e$, the edge derivatives $V_1 \cdots V_n u \in C_e^0(M, g)$ are again continuous up the boundary for any $n \leq k$. We put

$$\|u\|_k := \sup_M |u| + \sup_M |V_1 u| + \cdots + \sup_M |V_1 \cdots V_k u| < \infty.$$

- (ii) The Hölder space $C_e^\alpha(M, g)$, $\alpha \in (0, 1)$, consists of continuous functions u such that

$$\|u\|_\alpha := \sup_M |u| + \sup_M \frac{|u(p) - u(q)|}{d(p, q)^\alpha} < \infty,$$

where $d(p, q)$ represents the distance between $p, q \in M$ with respect to the Riemannian metric g . Note that in the local neighbourhood of the edge these distances are uniformly equivalent to

$$d((x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}))^2 \approx |x - \tilde{x}|^2 + |y - \tilde{y}|^2 + (x + \tilde{x})^2 |z - \tilde{z}|^2,$$

if g is incomplete, and

$$d((x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}))^2 \approx \frac{|x - \tilde{x}|^2 + |y - \tilde{y}|^2}{(x + \tilde{x})^2} + |z - \tilde{z}|^2,$$

if g is complete.

In the case of incomplete edge metrics it also makes sense to restrict the Banach space of continuous functions to those which are fibrewise constant at $x = 0$. This is precisely the space of continuous functions $\mathcal{C}_e^0(M, g)$ with respect to the topology on M induced by the Riemannian metric g . The corresponding space of continuous k -times edge-differentiable functions shall be denoted by $\mathcal{C}_e^k(M, g)$, with

$$\mathcal{C}_e^k(M, g) := \{u \in C_e^k(M, g) \mid V_1 \cdots V_j u \in \mathcal{C}_e^0(M, g) \text{ for any } V_i \in \mathcal{V}_e\},$$

which is a Banach subspace of $C_e^k(M, g)$. For the Hölder space with fractional differentiability we set $\mathcal{C}_e^\alpha(M, g) := C_e^\alpha(M, g)$. Note that similar considerations are possible in the setup of complete edge metrics, but in that case these do not lead to a refinement of the regularity statement.

Our estimates with respect to these spaces are as follows, where a precise definition of the heat operator is given in §2.

Theorem 1.5. *Let (M, g) be an m -dimensional Riemannian manifold with a feasible incomplete non-iterated edge metric. Let $e^{-t\Delta}$ denote the heat operator corresponding to the Friedrichs extension Δ of the associated Laplacian on (M, g) . Then $e^{-t\Delta}$ is bounded for any fixed $t \in \mathbb{R}^+$ with estimates of the form*

$$(1.0.1) \quad e^{-t\Delta} : C_e^k(M, g) \rightarrow C_e^{k+n}(M, g), \quad \|e^{-t\Delta} f\|_{C_e^{k+n}} \leq C t^{-n/2} \|f\|_{C_e^k},$$

$$(1.0.2) \quad e^{-t\Delta} : C_e^\alpha(M, g) \rightarrow C_e^2(M, g), \quad \|e^{-t\Delta} f\|_{C_e^2} \leq C t^{-1+\alpha/2} \|f\|_{C_e^\alpha}.$$

Precisely the same estimates hold if $C_e^(M, g)$ is replaced by $\mathcal{C}_e^*(M, g)$.*

For complete edges the result is similar, although the heat operator acts between weighted edge spaces in this case. A function u is in a weighted edge space, denoted $u \in x^w C_e^k(M, g)$, if and only if $u = x^w v$, with $v \in C_e^k(M, g)$.

Theorem 1.6. *Let (M, g) be an m -dimensional Riemannian manifold with a feasible complete non-iterated edge metric. Let $e^{-t\Delta}$ denote the heat operator of the unique self-adjoint extension Δ of the associated Laplacian on (M, g) . Then for any $w \in \mathbb{R}$, $e^{-t\Delta}$ is bounded for any fixed $t \in \mathbb{R}^+$ with estimates of the form*

$$(1.0.3) \quad e^{-t\Delta} : x^w C_e^k(M, g) \rightarrow x^w C_e^{k+n}(M, g), \quad \|e^{-t\Delta} f\|_{x^w C_e^{k+n}} \leq C t^{-n/2} \|f\|_{x^w C_e^k},$$

$$(1.0.4) \quad e^{-t\Delta} : x^w C_e^\alpha(M, g) \rightarrow x^w C_e^2(M, g), \quad \|e^{-t\Delta} f\|_{x^w C_e^2} \leq C t^{-1+\alpha/2} \|f\|_{x^w C_e^\alpha}.$$

An immediate observation from these estimates is that after taking into account the geometry of the underlying space, the mapping properties of the heat operator resemble the well-known behaviour on compact manifolds. As a particular consequence of our results, we establish short-time existence for solutions to certain semilinear parabolic equations on manifolds with edge metrics. Applications of such equations, including the reaction-diffusion equation, may be found in [CH98].

Corollary 1.7. *Let (M, g) be an m -dimensional Riemannian manifold with a feasible edge metric. Let Δ denote the Friedrichs extension of the Laplacian on (M, g) for an incomplete edge metric, and the unique self-adjoint extension of the Laplacian for a complete edge metric. Let the pair (X, Y) denote either $(C_e^2(M, g), C_e^\alpha(M, g))$ for $\alpha \in (0, 1)$ or*

- (i) $(\mathcal{C}_e^{k+1}(M, g), \mathcal{C}_e^k(M, g)), k \in \mathbb{N}$, if g is an incomplete edge metric;
- (ii) $(C_e^{k+1}(M, g), C_e^k(M, g)), k \in \mathbb{N}$, if g is a complete edge metric.

For $u \in X$, let $Q : X \rightarrow Y$ denote a locally Lipschitz inhomogeneous term that depends on at most one edge derivative of u . Then the initial value problem

$$\frac{\partial u}{\partial t} + \Delta u + Q(u, Vu) = 0, \quad u(0) = f \in Y,$$

admits a solution $u \in C([0, T]; X)$ on some time interval $[0, T]$ for $T > 0$.

Our discussion of the heat operator in the edge setup reviews and generalizes the work of Jeffres and Loya in [JL03] and [JL04], where the case $\dim B = 0$ was considered. Their work was based on the heat kernel analysis by Mooers [Moo96] in the incomplete (conical) case and by Melrose [MEL93] in the complete (b-cylindrical) setup. The analysis of the heat operator in the presence of incomplete singularities was initiated by Cheeger [CHE83], with major contributions by Brüning and Seeley [BS87], [BS91], Lesch [LES97], Melrose [MEL93] and Mazzeo [MAZ91], to select a few. Related questions on regularity properties of solutions to parabolic equations in the singular setup have been studied in [CSS02], [LI07] and [LOY01]. Most recently, a study of the inhomogeneous Cauchy problem on manifolds with incomplete conical metrics has been presented by Behrndt [BEH11].

The paper is organized as follows. In §2 we review the asymptotic properties of the heat kernel as a polyhomogeneous distribution on the appropriate blowup of the heat space. We discuss the incomplete and complete cases separately since the asymptotic properties are different. In §3 we apply the asymptotics of the heat kernel to derive the mapping properties of the heat operator and hence the regularity of solutions to the heat equation, carefully estimating the corresponding integral in various regions of the heat-space blowup. Finally, in §4 we explain how these mapping properties yield short-time existence for solutions to certain semilinear parabolic equations.

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2. ASYMPTOTICS OF THE HEAT KERNEL ON EDGE MANIFOLDS

Recall that we let (M^m, g) be a Riemannian manifold with an edge at B^b and a feasible edge metric g , where M is the interior of a compact manifold \overline{M} with boundary. This boundary ∂M is the total space of a fibration $\phi : \partial M \rightarrow B$, with fibre F^f . The local coordinates on \overline{M} near ∂M are given by (x, y, z) , where (y) are local coordinates on B lifted to ∂M and then extended inwards, and (z) restrict to local coordinates on F along each fibre of ∂M . Note that $m = 1 + b + f$.

Denote by Δ the Friedrichs extension of the Laplacian on (M, g) in the case of an incomplete edge, and the unique self-adjoint extension of the Laplacian in the case of a complete edge. The heat operator of Δ , denoted $e^{-t\Delta}$, solves the homogeneous heat problem

$$\begin{cases} (\partial_t + \Delta_g)u(x, y, z, t) &= 0 \\ u(x, y, z, 0) &= f(x, y, z), \end{cases}$$

with $u = e^{-t\Delta}f$. It is an integral operator

$$(2.0.1) \quad e^{-t\Delta}f(p) = \int_M H(t, p, \tilde{p}) f(\tilde{p}) \, \text{dvol}_g(\tilde{p}),$$

with the heat kernel H being a distribution on the heat space $M_h^2 := \mathbb{R}^+ \times \overline{M}^2$; local coordinates in the singular neighbourhood are given by $(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}))$, where (x, y, z) and $(\tilde{x}, \tilde{y}, \tilde{z})$ are coordinates on the two copies of M near the edge.

The heat kernel H has non-uniform behaviour at the diagonal D and at the submanifold

$$A = \begin{cases} \{(t, (0, y, z), (0, \tilde{y}, \tilde{z})) \in \mathbb{R}^+ \times (\partial M)^2 : t = 0, y = \tilde{y}\}, & \text{if } M \text{ is incomplete edge,} \\ \{(t, (0, y, z), (0, \tilde{y}, \tilde{z})) \in \mathbb{R}^+ \times (\partial M)^2 : y = \tilde{y}\}, & \text{if } M \text{ is complete edge.} \end{cases}$$

The asymptotic behaviour of H near these submanifolds of M_h^2 depends on the angle of approach to these submanifolds, and is crucial for estimation of the integral (2.0.1). This dependence on the angle in the asymptotics of the heat kernel is conveniently treated by introducing polar coordinates around A and D . Geometrically this corresponds to appropriate blowups of the heat space M_h^2 , distinct for the incomplete and complete cases, such that the corresponding heat kernel lifts to a polyhomogeneous distribution on that blowup space, in the sense of the following definition.

Definition 2.1. Let \mathfrak{W} be a manifold with corners, with all boundary faces embedded, and $\{(H_i, \rho_i)\}_{i=1}^N$ an enumeration of its boundaries and the corresponding defining functions. For any multi-index $b = (b_1, \dots, b_N) \in \mathbb{C}^N$ we write $\rho^b = \rho_1^{b_1} \dots \rho_N^{b_N}$. Denote by $\mathcal{V}_b(\mathfrak{W})$ the space of smooth vector fields on \mathfrak{W} which lie tangent to all boundary faces. A distribution w on \mathfrak{W} is said to be conormal if $w \in \rho^b L^\infty(\mathfrak{W})$ for some $b \in \mathbb{C}^N$ and $V_1 \dots V_\ell w \in \rho^b L^\infty(\mathfrak{W})$ for all $V_j \in \mathcal{V}_b(\mathfrak{W})$ and for every $\ell \geq 0$. An index set $E_i = \{(\gamma, p)\} \subset \mathbb{C} \times \mathbb{N}$ satisfies the following hypotheses:

- (i) $\text{Re}(\gamma)$ accumulates only at $+\infty$;
- (ii) for each γ there exists $P_\gamma \in \mathbb{N}_0$ such that $(\gamma, p) \in E_i$ for $p \leq P_\gamma < \infty$;
- (iii) if $(\gamma, p) \in E_i$, then $(\gamma + j, p') \in E_i$ for all $j \in \mathbb{N}$ and $0 \leq p' \leq p$.

An index family $E = (E_1, \dots, E_N)$ is an N -tuple of index sets. Finally, we say that a conormal distribution w is polyhomogeneous on \mathfrak{W} with index family E , denoted $w \in \mathcal{A}_{\text{phg}}^E(\mathfrak{W})$, if w is conormal and if near each H_i we have

$$w \sim \sum_{(\gamma, p) \in E_i} a_{\gamma, p} \rho_i^\gamma (\log \rho_i)^p, \text{ as } \rho_i \rightarrow 0,$$

with coefficients $a_{\gamma, p}$ conormal on H_i and polyhomogeneous with index E_j at any $H_i \cap H_j$.

For more on polyhomogeneous distributions and other relevant background material, we refer the reader to the classical references [MEL93] and [MAZ91], as well as the excellent introduction to the b -calculus by Grieser [GR99].

2.1. Heat kernel on spaces with incomplete edge metrics. To obtain the correct blowup of M_h^2 in the case of an incomplete edge metric, one first does a parabolic blowup of the submanifold A defined above. The resulting heat space $[M_h^2, A]$ is defined as the union of $M_h^2 \setminus A$ with the interior spherical normal bundle of A in M_h^2 . The blowup $[M_h^2, A]$ is endowed with the unique minimal differential structure with respect to which smooth functions in the interior of M_h^2 and polar coordinates on M_h^2 around A are smooth. This blowup introduces four new boundary hypersurfaces, which we denote by ff (the front face), rf (the right face), lf (the left face) and tf (the temporal face).

The actual heat-space blowup \mathcal{M}_h^2 is obtained by a parabolic blowup of $[M_h^2, A]$ along the diagonal

$$D := \{t = 0, x = \tilde{x}, y = \tilde{y}, z = \tilde{z}\} \subset M_h^2,$$

lifted to a submanifold of $[M_h^2, A]$. The resulting blowup \mathcal{M}_h^2 is defined as before by cutting out the submanifold and replacing it with its spherical normal bundle; a new boundary hypersurface, the temporal diagonal (td), appears. The blowup \mathcal{M}_h^2 is a manifold with boundaries and corners as depicted in Figure 1.

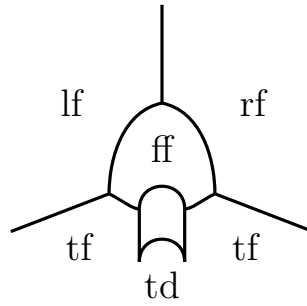


FIGURE 1. Heat-space blowup \mathcal{M}_h^2 for incomplete edge metrics

The appropriate projective coordinates on \mathcal{M}_h^2 are given as follows. Near the top corner of ff away from tf the projective coordinates are given by

$$\rho = \sqrt{t}, \quad \xi = \frac{x}{\rho}, \quad \tilde{\xi} = \frac{\tilde{x}}{\rho}, \quad u = \frac{y - \tilde{y}}{\rho}, \quad y, \quad z, \quad \tilde{z},$$

where in these coordinates $\rho, \xi, \tilde{\xi}$ are the defining functions of the faces ff, rf and lf, respectively. For the bottom corner of ff near lf, the projective coordinates are given by

$$\tau = \frac{t}{x^2}, \quad s = \frac{\tilde{x}}{x}, \quad u = \frac{y - \tilde{y}}{x}, \quad x, \quad y, \quad z, \quad \tilde{z},$$

where in these coordinates τ, s, x are the defining functions of tf, lf and ff, respectively. For the bottom corner of ff near rf the projective coordinates are obtained by interchanging the roles of x and \tilde{x} . The projective coordinates on \mathcal{M}_h^2 near the top of td away from tf are given by

$$\eta = \sqrt{\tau}, \quad S = \frac{1-s}{\eta}, \quad U = \frac{u}{\eta}, \quad Z = \frac{z - \tilde{z}}{\eta}, \quad x, \quad y, \quad z.$$

In these coordinates tf is the face in the limit $|(S, U, Z)| \rightarrow \infty$, and ff and td are defined by \tilde{x} and η , respectively. The blowup heat space \mathcal{M}_h^2 is related to the original heat space M_h^2 via the obvious ‘blow-down map’ $\beta : \mathcal{M}_h^2 \rightarrow M_h^2$, which in local coordinates is simply the coordinate change back to $(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}))$.

We can now state the asymptotic properties of the heat kernel as a polyhomogeneous distribution on the blowup \mathcal{M}_h^2 . These have been established by Mazzeo and the third author in [MV10]. The Friedrichs extension of the Laplace operator is a self-adjoint operator in the Hilbert space $L^2(M, \text{vol}(g))$, and due to the specific form of the incomplete edge metric we have $\text{vol}(g) = x^f dx \wedge \text{vol}^{\partial M}(x)$ near the edge singularity. Hence one needs to take the singular factor of x^f of the volume into account in order to obtain the correct asymptotic behaviour.

Theorem 2.2. [MV10] *The heat kernel H lifts via the blowdown map β to a polyhomogeneous distribution β^*H on \mathcal{M}_h^2 , with asymptotic expansion of leading order $(-\dim M)$ at the front face ff and at the diagonal face td, smooth at the left and right boundary faces, and vanishing to infinite order at tf.*

The heat kernel asymptotics were derived in [MV10] under a unitary rescaling transformation in the singular neighbourhood $U = (0, 1) \times \partial M \subset M$ of the edge singularity, similar to Brüning-Seeley [BS88]

$$\Phi : C_{\text{pct}}^\infty(U) \rightarrow C_{\text{pct}}^\infty(U), \quad u \mapsto x^{f/2}u,$$

such that the transformed Laplacian $\Phi \circ \Delta \circ \Phi^{-1}$ is self-adjoint in $L^2(U, dx \wedge \text{vol}^{\partial M}(x))$. In particular the leading order of the heat kernel asymptotics in [MV10] refer to the heat operator as an integral operator with respect to the volume $dx \wedge \text{vol}^{\partial M}(x)$. This accounts for the difference in powers of the defining functions between Theorem 2.2 and [MV10].

To see this explicitly, we compare the actions of the heat operators with and without the unitary rescaling transformation. We may assume for simplicity that the heat kernel is compactly supported in $U^2 \times \mathbb{R}^+$ and find

$$\begin{aligned} \exp(-t\Phi \circ \Delta \circ \Phi^{-1})u &= \int_U \exp(-t\Phi \circ \Delta \circ \Phi^{-1})(x, y, z, \tilde{x}, \tilde{y}, \tilde{z})u(\tilde{x}, \tilde{y}, \tilde{z})d\tilde{x} \, d\text{vol}_{\partial M}(\tilde{x}) \\ &= \int_U x^{f/2} \exp(-t\Delta)(x, y, z, \tilde{x}, \tilde{y}, \tilde{z})\tilde{x}^{-f/2}u(\tilde{x}, \tilde{y}, \tilde{z})\tilde{x}^f d\tilde{x} \, d\text{vol}_{\partial M}(\tilde{x}). \end{aligned}$$

Consequently we deduce the following relation between the heat kernels:

$$\exp(-t\Phi \circ \Delta \circ \Phi^{-1})(x, y, z, \tilde{x}, \tilde{y}, \tilde{z}) = (x\tilde{x})^{f/2} \exp(-t\Delta)(x, y, z, \tilde{x}, \tilde{y}, \tilde{z}).$$

One may now check in projective blowup coordinates that the lifts of the heat kernels to the blowup heat space are related as follows

$$(2.1.1) \quad \beta^* \exp(-t\Phi \circ \Delta \circ \Phi^{-1}) = \rho_{\text{ff}}^f (\rho_{\text{rf}} \rho_{\text{lf}})^{f/2} \beta^* \exp(-t\Delta),$$

where, e.g., ρ_{rf} is a defining function for the right face. The rescaled heat kernel $\exp(-t\Phi \circ \Delta \circ \Phi^{-1})$ has been discussed in [MV10] and lifts to a density on the heat-space blowup with asymptotic expansion of leading order $(-1-b)$ at ff, of leading order $(f/2)$ at lf and rf, and of leading order $(-\dim M)$ at td. Employing the relation (2.1.1) we arrive at the actual statement of Theorem 2.2.

Remark 2.3. A similar situation arises in the special case of a cone, where the edge B is collapsed to a point. This setup was considered by Mooers in [Moo96], where the heat kernel was considered under a unitary rescaling transformation as well. Without the rescaling one needs to take the singular factor x^f of the volume $x^f dx \wedge \text{vol} g^{\partial M}(x)$ into account, leading to different asymptotic behaviour. It appears this was not done in [JL03].

We will also need a more refined result on the asymptotic behaviour of the heat kernel at the right boundary face of \mathcal{M}_h^2 .

Proposition 2.4. *Let (M^m, g) be a feasible edge space. Then the lift $\beta^* H$ is a polyhomogeneous distribution on \mathcal{M}_h^2 , and letting s denote a defining function of rf, the expansion of $\beta^* H$ takes the form*

$$\beta^* H \sim s^0 a_0(\beta^* H) + O(s^{\gamma_0}) \text{ as } s \rightarrow 0$$

for some $\gamma_0 > 0$, where $a_0(\beta^* H)$ is constant in the first fibre variable z .

Proof. Recall the heat kernel construction in [MV10]. The initial approximative parametrix for the solution operator of $\mathcal{L} = (\partial_t + \Delta)$ is constructed in [MV10] by solving the heat equation to first order at the front face ff of \mathcal{M}_h^2 . The restriction of the lift $\beta^*(t\mathcal{L})$ to ff is called the normal operator $N_{\text{ff}}(t\mathcal{L})$ at the front face and feasibility of g , more precisely Definition 1.2 (i), ensures that $N_{\text{ff}}(t\mathcal{L})$ is given in projective coordinates $(\tau, s, u, z, \tilde{x}, \tilde{y}, \tilde{z})$ by

$$N_{\text{ff}}(t\mathcal{L}) = \tau(\partial_\tau - \partial_s^2 - fs^{-1}\partial_s + s^{-2}\Delta_{F,z} + \Delta_u^{\mathbb{R}^b}) =: \tau(\partial_\tau + \Delta_s^{\mathcal{C}(F)} + \Delta_u^{\mathbb{R}^b}).$$

Here $\mathcal{C}(F) = \mathbb{R}_s^+ \times F_z$ denotes the model cone. Since $N_{\text{ff}}(t\mathcal{L})$ does not involve derivatives with respect to $(\tilde{x}, \tilde{y}, \tilde{z})$, it acts tangentially to the fibres of the front face. Searching for an initial parametrix H_0 , we solve the heat equation to first order at the front face, and want

$$N_{\text{ff}}(t\mathcal{L} \circ H_0) = N_{\text{ff}}(t\mathcal{L}) \circ N_{\text{ff}}(H_0) = 0,$$

which is the heat equation on the model edge $\mathcal{C}(F) \times \mathbb{R}_u^b$. Consequently, the initial parametrix H_0 is defined by choosing $N_{\text{ff}}(H_0)$ to equal the fundamental solution for the

heat operator $N_{\text{ff}}(t\mathcal{L})$, and extending $N_{\text{ff}}(H_0)$ trivially to a neighbourhood of the front face. Using the projective coordinates $(\tau, s, u, z, \tilde{x}, \tilde{y}, \tilde{z})$ near ff we have

$$(2.1.2) \quad N_{\text{ff}}(H_0) := H^{\mathcal{C}(F)}(\tau, s, z, 1, \tilde{z}) H^{\mathbb{R}^b}(\tau, u, 0),$$

where $H^{\mathbb{R}^b}$ denotes the euclidean heat kernel on \mathbb{R}^b and $H^{\mathcal{C}(F)}$ is the heat kernel for the exact cone $\mathcal{C}(F)$, as studied in [CHE83], [LES97] and [MOO96]. Thus, the index set $E \subset [0, \infty) \times \{0\}$ for the asymptotic behaviour of H_0 at the right and left boundary faces is given in terms of the indicial roots γ for the Laplacian $\Delta_s^{\mathcal{C}(F)}$ on the exact cone, i.e.,

$$H_0 \sim \sum_{\gamma \in E} s^\gamma a_\gamma(H_0) \text{ as } s \rightarrow 0,$$

with the leading coefficient $a_0(H_0)$ being harmonic on fibres and hence constant in z . Feasibility of g , more precisely Definition 1.2 (ii), ensures that the indicial roots are independent of the base point $b \in Y$; hence the index set E is discrete and H_0 is indeed polyhomogeneous on \mathcal{M}_h^2 . The error of the initial parametrix H_0 is given by

$$\beta^*(t\mathcal{L})H_0 = \left[\beta^*(t\Delta_g) - \tau\Delta_s^{\mathcal{C}(F)} - \tau\Delta_u^{\mathbb{R}^b} \right] H_0 =: P_0.$$

The asymptotic expansion of H_0 as $s \rightarrow 0$ starts with $s^0 a_0$ with no $s^0(\log s)^k, k \geq 1$ terms. The leading term $s^0 a_0$ is annihilated by the edge vector fields $s\partial_s$ and ∂_z since a_0 is constant in z , and its order is raised by $s\partial_u$. Consequently, since $\left[\beta^*(t\Delta_g) - \tau\Delta_s^{\mathcal{C}(F)} - \tau\Delta_u^{\mathbb{R}^b} \right]$ is of higher order in s , we find

$$P_0 \sim \sum_{l=0}^{\infty} s^l a_l(P_0) + \sum_{\gamma \in E^*} \sum_{l=0}^{\infty} s^{\gamma-1+l} a_{\gamma,l}(P_0) \text{ as } s \rightarrow 0,$$

where $E^* := E \setminus \{0\}$.

The next step in the construction of the heat kernel involves adding a kernel J to H_0 , such that the new error term is vanishing to infinite order at rf. In order to eliminate the term $s^\gamma a_\gamma$ in the asymptotic expansion of P_0 at rf, we only need to solve

$$(2.1.3) \quad (-\partial_s^2 - fs^{-1}\partial_s + s^{-2}\Delta_{F,z})\omega = s^\gamma(\tau^{-1}a_\gamma).$$

This is because all other terms in the expansion of $t\mathcal{L}$ at rf lower the exponent in s by at most one, while the indicial part lowers the exponent by two. The variables $(\tau, u, \tilde{x}, \tilde{y}, \tilde{z})$ enter the equation only as parameters. The equation on the model cone is solved via the Mellin transform and the solution ω is polyhomogeneous in all variables including parameters, and is of leading order $(\gamma + 2)$. Hence $H_1 = H_0 + J$ expands near rf as

$$H_1 \sim s^0 a_0(H_1) + \sum_{l=2}^{\infty} s^l a_l(H_1) + \sum_{\gamma \in E^*} \sum_{l=0}^{\infty} s^{\gamma+l} a_{\gamma,l}(H_1) \text{ as } s \rightarrow 0,$$

where $a_0(H_1) \equiv a_0(H_0)$ is constant in z .

In the following correction steps the exact heat kernel is obtained from H_1 first by improving the error near td and then by an iterative correction procedure, adding terms of the form $H_1 \circ (P_1)^k$, where $P_1 := t\mathcal{L}H_1$ is vanishing to infinite order at rf and td. Since

the leading coefficient $a_0(H_1)$ is constant in z , we deduce $\partial_z H_1 \sim O(s^{\gamma_0})$ as $s \rightarrow 0$, where $\gamma_0 := \min\{2, \gamma \in E^*\}$. Consequently, we also have $\partial_z H_1 \circ (P_1)^k \sim O(s^{\gamma_0})$ as $s \rightarrow 0$. We find

$$\beta^* H \sim s^0 a_0(\beta^* H) + \sum_{l=2}^{\infty} s^l a_l(\beta^* H) + \sum_{\gamma \in E^*} \sum_{l=0}^{\infty} s^{\gamma+l} a_{\gamma,l}(\beta^* H) \text{ as } s \rightarrow 0,$$

where the leading coefficient $a_0(\beta^* H) \equiv a_0(H_0)$ is still constant in z . \square

2.2. Heat kernel on spaces with a complete edge metric. We next describe the blowup space and asymptotic structure for the heat kernel of the Laplacian of a complete edge metric. The blowup proceeds exactly as in §2.1 except that in the first step we blow up the heat space M_h^2 along the submanifold

$$A = \{(t, (0, y, z), (0, \tilde{y}, \tilde{z})) \in \mathbb{R}^+ \times (\partial M)^2 : y = \tilde{y}\} \subset M_h^2.$$

The final heat-space blowup \mathcal{M}_h^2 for a complete edge metric is depicted in Figure 2.

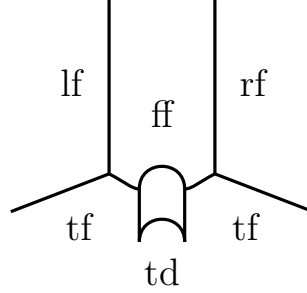


FIGURE 2. Heat-space blowup \mathcal{M}_h^2 for complete edge metrics

We now describe some coordinates on \mathcal{M}_h^2 . Near the left hand corner where the ff, tf and lf meet we may use projective coordinates for \mathcal{M}_h^2 , valid away from $x = 0$:

$$(2.2.1) \quad \tau = \sqrt{t}, \quad s = \frac{\tilde{x}}{x}, \quad u = \frac{y - \tilde{y}}{x}, \quad x, \quad y, \quad z, \quad \tilde{z}.$$

Observe that in these coordinates, s and x are defining functions for lf and ff, respectively. Near the top of the blown up diagonal, where td and ff meet, and away from tf we may use the coordinates

$$(2.2.2) \quad \eta = \sqrt{t}, \quad S = \frac{x - \tilde{x}}{x\sqrt{t}}, \quad U = \frac{y - \tilde{y}}{x\sqrt{t}}, \quad Z = \frac{z - \tilde{z}}{\sqrt{t}}, \quad x, \quad y, \quad z.$$

In these coordinates, η is a defining function for td and x is a defining function for ff. By abuse of notation, we will denote the blowdown map with respect to any coordinates by β , and will denote the composition of β with projections onto the left and right factors in M_h^2 by $\beta_L : \mathcal{M}_h^2 \rightarrow \mathbb{R}^+ \times M$ and $\beta_R : \mathcal{M}_h^2 \rightarrow M$, respectively.

The asymptotic structure of the heat kernel of the Laplacian of a complete edge metric was described by Pierre Albin in [ALB07].

Theorem 2.5. [ALB07] *The heat kernel H lifts via the blowdown map β to a polyhomogeneous distribution β^*H on \mathcal{M}_h^2 with asymptotic expansion smooth up to the boundary of the front face ff , with leading order $(-\dim M)$ at the temporal diagonal face td , and vanishing to infinite order at left, right and temporal boundary faces, lf , rf , tf , respectively.*

Albin worked in the context of the heat operator acting on half-densities. We dispense with half-densities and instead regard the heat kernel as acting on functions (with compact support near the edge) by:

$$e^{-t\Delta}f(x, y, z) = \int_M H(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) f(\tilde{x}, \tilde{y}, \tilde{z}) \tilde{x}^{-b-1} d\tilde{x} \, \text{dvol}_{\partial M}(\tilde{x}),$$

where $\text{dvol}_{\partial M}$ denotes the volume with respect to the metric $\phi^*g^B(x) + g^F(x)$ on ∂M . This accounts for the difference in powers of the defining functions from [ALB07]. We sketch how to apply Albin's result to get the asymptotics of the heat kernel acting on functions instead of half-densities.

We identify functions and half-densities on $\mathbb{R}^+ \times M^2$, on the factor $\mathbb{R}^+ \times M$, and on the factor M by¹

$$f(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) \leftrightarrow f(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) x^{-\frac{(b+1)}{2}} (\tilde{x})^{-\frac{(b+1)}{2}} |dx dy dz d\tilde{x} d\tilde{y} d\tilde{z} dt|^{1/2},$$

$$f(t, (x, y, z)) \leftrightarrow f(t, (x, y, z)) x^{-\frac{(b+1)}{2}} |dx dy dz dt|^{1/2},$$

$$f(x, y, z) \leftrightarrow f(x, y, z) x^{-\frac{(b+1)}{2}} |dx dy dz|^{1/2}.$$

As discussed in [ALB07], an element of the heat calculus $A \in \Psi_{e, \text{Heat}}^{2,0}$ has an integral kernel that may be written as $\rho_{\text{td}}^{-\frac{(b+f)}{2}} \rho_{\text{ff}}^{-\frac{(b+1)}{2}} k \cdot \nu$, where ρ_{face} is a defining function for the specified face; k is a function that vanishes to infinite order at lf , rf , and tf , and is smooth up to the boundary at td and ff ; and ν is a smooth section of $\Omega^{1/2}(\mathcal{M}_h^2)$. Such an operator $A \in \Psi_{e, \text{Heat}}^{2,0}$ acts on half-densities by

$$\begin{aligned} (2.2.3) \quad & A f(t, (x, y, z)) x^{-\frac{(b+1)}{2}} |dx dy dz dt|^{1/2} \\ &= (\beta_L)_* \left(\rho_{\text{td}}^{-\frac{(b+f)}{2}} \rho_{\text{ff}}^{-\frac{(b+1)}{2}} k \nu \cdot (\beta_R)^* (f(\tilde{x}, \tilde{y}, \tilde{z})) (\tilde{x})^{-\frac{(b+1)}{2}} |d\tilde{x} d\tilde{y} d\tilde{z}|^{1/2} \right). \end{aligned}$$

To relate these half-densities, let us work in coordinates (2.2.2). We may take $\nu = |dS dU dZ dx dy dz d\eta|^{1/2}$. Pulling back our standard half-density on $M^2 \times \mathbb{R}^+$, we find

$$\begin{aligned} (2.2.4) \quad & \beta^* (x^{-\frac{(b+1)}{2}} (\tilde{x})^{-\frac{(b+1)}{2}} |dx dy dz d\tilde{x} d\tilde{y} d\tilde{z} dt|^{1/2}) \\ &= x^{-\frac{(b+1)}{2}} (x(1-\eta S))^{-\frac{(b+1)}{2}} (-1)^{\frac{b+f+1}{2}} \sqrt{2} x^{\frac{b+1}{2}} \eta^{\frac{b+f+2}{2}} |dS dU dZ dx dy dz d\eta|^{1/2} \\ &= \sqrt{2} (-1)^{\frac{m}{2}} (1-\eta S)^{-\frac{(b+1)}{2}} x^{-\frac{(b+1)}{2}} \eta^{\frac{b+f+2}{2}} \nu. \end{aligned}$$

¹These densities have a factor of $\sqrt{\det g}$ in appropriate places. In what follows we omit smooth factors.

Since $1 - \eta S = \frac{\tilde{x}}{x}$ and we are working near the intersection of td and ff , the factor $\sqrt{2}(-1)^{\frac{m}{2}}(1 - \eta S)^{-\frac{(b+1)}{2}}$ is smooth and uniformly bounded, so we absorb it into k . Denote $\sigma_R := \beta_R^*((\tilde{x})^{-\frac{(b+1)}{2}}|d\tilde{x}d\tilde{y}d\tilde{z}|^{1/2})$. Using (2.2.3) and (2.2.4) we compute

$$\begin{aligned}
 & [A f](t, (x, y, z)) x^{-\frac{(b+1)}{2}} |dxdydzdt|^{1/2} \\
 &= (\beta_L)_* \left(\eta^{-\frac{(b+f)}{2}} x^{-\frac{(b+1)}{2}} k\nu \cdot (\beta_R)^*(f(\tilde{x}, \tilde{y}, \tilde{z})(\tilde{x})^{-\frac{(b+1)}{2}} |d\tilde{x}d\tilde{y}d\tilde{z}|^{1/2}) \right) \\
 &= (\beta_L)_* \left(\eta^{-\frac{(b+f)}{2}} x^{-\frac{(b+1)}{2}} k(\eta, (S, U, Z), (x, y, z)) \right. \\
 &\quad \cdot x^{\frac{(b+1)}{2}} \eta^{-\frac{(b+f+2)}{2}} \beta^*(x^{-\frac{(b+1)}{2}}(\tilde{x})^{-\frac{(b+1)}{2}} |dxdydzd\tilde{x}d\tilde{y}d\tilde{z}dt|^{1/2}) (\beta_R^* f) \sigma_R \Big) \\
 &= (\beta_L)_* \left(\eta^{-m} k(\eta, (S, U, Z), (x, y, z)) (\beta_R^* f) \right. \\
 &\quad \cdot \beta^*(x^{-\frac{(b+1)}{2}}(\tilde{x})^{-\frac{(b+1)}{2}} |dxdydzd\tilde{x}d\tilde{y}d\tilde{z}dt|^{1/2}) \sigma_R \Big) \\
 &= \left[\int H \left(t^{1/2}, \frac{x - \tilde{x}}{xt^{1/2}}, \frac{y - \tilde{y}}{xt^{1/2}}, \frac{z - \tilde{z}}{t^{1/2}}, x, y, z \right) f(\tilde{x}, \tilde{y}, \tilde{z}) \tilde{x}^{-(b+1)} d\tilde{x}d\tilde{y}d\tilde{z} \right] \\
 &\quad \cdot x^{-\frac{(b+1)}{2}} |dxdydzdt|^{1/2}.
 \end{aligned}$$

Canceling the half density from the first and last components of this equation gives the action of the heat kernel on functions, with the stated asymptotics.

2.3. Stochastic completeness of the heat kernel. Several of our arguments are simplified by the fact that the edge heat kernel is stochastically complete, i.e., satisfies

$$(2.3.1) \quad \int_M H(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) d\text{vol}_g = 1, \text{ for all } (x, y, z) \in M, t > 0.$$

In the incomplete case this is a consequence of uniqueness of solutions to the heat equation. More precisely, with Δ being a self-adjoint unbounded operator on the Hilbert space $L^2(M, \text{vol}(g))$ with domain $\mathcal{D}(\Delta)$, the solutions to the initial value problem

$$\partial_t u + \Delta u = 0, \quad u(0) = u_0 \in \mathcal{D}(\Delta) \subset L^2(M, \text{vol}(g))$$

are unique and in fact given by $u(t) = e^{-t\Delta} u_0 \in \mathcal{D}(\Delta)$ for any $t > 0$. Consequently, $u \equiv 1$ is the unique solution in $\mathcal{D}(\Delta)$ to the heat equation with initial value 1, given in terms of the heat operator by $u \equiv 1 = e^{-t\Delta} 1$. This is precisely (2.3.1).

In the complete edge case, $u \equiv 1$ does not lie in the Hilbert space $L^2(M, \text{vol}(g))$ and on general complete manifolds, solutions of the heat equation need not be unique. However by a result of Yau [Yau78] we have (2.3.1) on complete manifolds with Ricci curvature bounded from below; see [Hsu89] for a discussion of this and related results. In our setting, since an edge metric can be viewed at highest order as the product of an asymptotically hyperbolic metric and a compact metric, it is straightforward to check that the Ricci curvature is bounded from below.

3. MAPPING PROPERTIES OF THE HEAT KERNEL ON EDGE MANIFOLDS

3.1. Estimates on spaces with incomplete edge metrics. We prove Theorem 1.5. The main task in the proof is to check the estimates, as the continuity of $e^{-t\Delta} f$ and

its derivatives are easy to obtain by the dominated convergence theorem in each of the coordinate systems that will be discussed. The estimates in Theorem 1.5 are classical for the lift of the heat operator to \mathcal{M}_h^2 supported away from the front face. So we can assume without loss of generality that the lift of the heat kernel to the heat space blowup \mathcal{M}_h^2 is compactly supported in an open neighbourhood of the front face. Then the heat operator acts on functions as in (2.0.1) and the proof amounts to estimating $(k+n)$ edge derivatives V_e^{k+n} locally:

$$(3.1.1) \quad \int_M V_e^{k+n} H(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) f(\tilde{x}, \tilde{y}, \tilde{z}) \tilde{x}^f d\tilde{x} d\text{vol}_{\partial M}(\tilde{x}).$$

The estimates require a precise understanding of the asymptotic behaviour of H . Since the asymptotics of the heat kernel H near the front face are non-uniform, one needs to estimate the integral (3.1.1) for the integrand supported near the various corners and boundary faces of the heat space blowup \mathcal{M}_h^2 . We thus separate H into four components, with their lifts to the heat space blowup \mathcal{M}_h^2 compactly supported near each of the three corners of the front face and near the diagonal of \mathcal{M}_h^2 . This corresponds to different asymptotic regimes as the variables (t, x, \tilde{x}) approach zero, with the associated projective coordinates from §2.1 capturing the various asymptotic behaviour. We establish the estimates for the integral operators defined by each of the components separately.

Estimates near the lower left corner of the front face

We assume that the heat kernel H is compactly supported near the lower left corner of the front face. Its asymptotic behaviour is appropriately described in the projective coordinates

$$\tau = \frac{t}{x^2}, \quad s = \frac{\tilde{x}}{x}, \quad u = \frac{y - \tilde{y}}{x}, \quad x, \quad y, \quad z, \quad \tilde{z},$$

where τ, s , and x are the defining functions of tf , lf and ff , respectively. The coordinates are valid whenever (τ, s, u) are bounded as (t, x, \tilde{x}) approach zero. The edge vector fields obey the following transformation rules:

$$\beta^*(x\partial_x) = -2\tau\partial_\tau - s\partial_s - u\partial_u + x\partial_x, \quad \beta^*(x\partial_y) = \partial_u + x\partial_y, \quad \beta^*(\partial_z) = \partial_z.$$

Note that we abuse notation by writing $x\partial_x, x\partial_y$ vector fields on the right side of these expressions. By Theorem 2.2 we infer that for any $n \in \mathbb{N}_0$,

$$\beta^*(V_e^n H)(\tau, x, y, z, s, u, \tilde{z}) = x^{-n} G(\tau, x, y, z, s, u, \tilde{z}),$$

where G is bounded in its entries and infinitely vanishing at τ goes to zero, with the same compact support as β^*H . For the transformation rule of the volume form we compute

$$\beta^*(\tilde{x}^f d\tilde{x} d\text{vol}_{\partial M}(\tilde{x})) = h \cdot s^f x^m ds du d\tilde{z},$$

where h is a bounded distribution on \mathcal{M}_h^2 . Hence we can write for any $f \in C_e^k(M, g)$

$$\begin{aligned} V_e^n e^{-t\Delta} f(x, y, z) &= \int_M V_e^n H(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) f(\tilde{x}, \tilde{y}, \tilde{z}) \tilde{x}^f d\tilde{x} d\text{vol}_{\partial M}(\tilde{x}) \\ &= \int h \cdot G \cdot f(sx, y - ux, \tilde{z}) s^f ds du d\tilde{z}. \end{aligned}$$

Pulling out the supremum of f and estimating $h \cdot G$ against a constant, we find

$$|V_e^n e^{-t\Delta} f(x, y, z)| \leq C \|f\|_\infty \leq C \cdot \|f\|_{C_e^k},$$

Note that this estimate also accounts for the second statement of the theorem, since $C_e^\alpha(M, g) \subset C_e^0(M, g)$.

Estimates near the lower right corner of the front face.

Applying our argument for the left corner of the front face near the right corner appears to produce singular behaviour in powers of the defining function for the front face. However, the argument is not symmetric in the (x, y, z) versus $(\tilde{x}, \tilde{y}, \tilde{z})$ variables as these variables play different roles in the integration (2.0.1). Thus we may use a second change of variables to take advantage of the infinite order decay at the temporal face. To begin, we use the projective coordinates

$$\tilde{\tau} = \frac{t}{\tilde{x}^2}, \quad \tilde{s} = \frac{x}{\tilde{x}}, \quad \tilde{u} = \frac{y - \tilde{y}}{\tilde{x}}, \quad z, \quad \tilde{x}, \quad \tilde{y}, \quad \tilde{z},$$

where $\tilde{\tau}$, \tilde{s} , and \tilde{x} are the defining functions of tf, rf and ff, respectively. The coordinates are valid whenever $(\tilde{\tau}, \tilde{s}, \tilde{u})$ are bounded as (t, x, \tilde{x}) approach zero. The edge vector fields obey the following transformation rules:

$$\beta^*(x\partial_x) = \tilde{s}\partial_{\tilde{s}}, \quad \beta^*(x\partial_y) = \tilde{s}\partial_{\tilde{u}}, \quad \beta^*(\partial_z) = \partial_z.$$

By Theorem 2.2 we infer that for any $n \in \mathbb{N}_0$,

$$\beta^*(V_e^n H)(\tilde{\tau}, \tilde{s}, \tilde{u}, z, \tilde{x}, \tilde{y}, \tilde{z}) = \tilde{x}^{-m} G(\tilde{\tau}, \tilde{s}, \tilde{u}, z, \tilde{x}, \tilde{y}, \tilde{z}),$$

where G is bounded in its entries and vanishing to infinite order in $\tilde{\tau}$ as $\tilde{\tau}$ goes to zero, with the same compact support as β^*H . Of course G depends on the particular derivatives taken, but we will not express this in the notation.

In order to estimate (3.1.1) we again perform a change of variable, this time using the time variable in order to take advantage of the decay of the heat kernel at tf. This gives

$$\tilde{x}^f d\tilde{x} d\text{vol}_{\partial M}(\tilde{x}) = h \cdot \tilde{\tau}^{-1} \tilde{x}^{1+f+b} d\tilde{\tau} d\tilde{u} d\tilde{z} = h \cdot \tilde{\tau}^{-1} \tilde{x}^m d\tilde{\tau} d\tilde{u} d\tilde{z},$$

where h is a bounded distribution on \mathcal{M}_h^2 . Therefore for any $f \in C_e^k(M, g)$

$$\begin{aligned} V_e^n e^{-t\Delta} f(x, y, z) &= \int \tilde{x}^{-m} (\beta_* G)(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) f(\sqrt{t}/\sqrt{\tilde{\tau}}, y - \tilde{u}\sqrt{t}/\sqrt{\tilde{\tau}}, \tilde{z}) h \cdot \tilde{\tau}^{-1} \tilde{x}^m d\tilde{\tau} d\tilde{u} d\tilde{z} \\ &= \int \tilde{\tau}^{-1} (\beta_* G)(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) f(\sqrt{t}/\sqrt{\tilde{\tau}}, y - \tilde{u}\sqrt{t}/\sqrt{\tilde{\tau}}, \tilde{z}) h \cdot d\tilde{\tau} d\tilde{u} d\tilde{z}, \end{aligned}$$

where G is vanishing to infinite order in $\tilde{\tau}$ and bounded in its other entries. We conclude

$$\|V_e^n e^{-t\Delta} f(x, y, z)\|_{C_e^0} \leq C \|f\|_{C_e^0}.$$

Further, we find from the expressions for the edge vector fields that

$$\|e^{-t\Delta} f(x, y, z)\|_{C_e^k} \leq C \|f\|_{C_e^0} \leq C \|f\|_{C_e^k}$$

and

$$\|e^{-t\Delta} f(x, y, z)\|_{C_e^2} \leq C \|f\|_{C_e^0} \leq C \|f\|_{C_e^\alpha}.$$

Estimates near the top corner of the front face

The asymptotic behaviour of the heat kernel H , compactly supported near the top corner of ff , is appropriately described in the projective coordinates

$$\rho = \sqrt{t}, \quad \xi = \frac{x}{\rho}, \quad \tilde{\xi} = \frac{\tilde{x}}{\rho}, \quad u = \frac{y - \tilde{y}}{\rho}, \quad y, \quad z, \quad \tilde{z},$$

where ρ, ξ , and $\tilde{\xi}$ are the defining functions of the faces ff , rf and lf , respectively. The coordinates are valid whenever $(\rho, \xi, \tilde{\xi})$ are bounded as (t, x, \tilde{x}) approach zero. The edge vector fields obey the following transformation rules:

$$\beta^*(x\partial_x) = \xi\partial_\xi, \quad \beta^*(x\partial_y) = \xi\partial_u + \rho\xi\partial_y, \quad \beta^*(\partial_z) = \partial_z.$$

Hence by Theorem 2.2 we infer that for any $n \in \mathbb{N}_0$,

$$\beta^*(V_e^n H) \left(\rho, \xi, y, z, \tilde{\xi}, u, \tilde{z} \right) = \rho^{-m} G(\rho, \xi, y, z, \tilde{\xi}, u, \tilde{z}),$$

where G is bounded in its entries and has the same compact support as β^*H . For the transformation rule of the volume form we compute

$$\beta^*(\tilde{x}^f d\tilde{x} \, \text{dvol}_{\partial\text{M}}(\tilde{x})) = h \cdot \tilde{\xi}^f \rho^{1+f+b} d\tilde{\xi} du d\tilde{z},$$

where h is a bounded distribution on \mathcal{M}_h^2 . Hence we can write for any $f \in C_e^k(M, g)$

$$\begin{aligned} V_e^n e^{-t\Delta} f(x, y, z) &= \int_M V_e^n H(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) f(\tilde{x}, \tilde{y}, \tilde{z}) \tilde{x}^f d\tilde{x} \, \text{dvol}_{\partial\text{M}}(\tilde{x}) \\ &= \int h \cdot G \cdot f(\rho\tilde{\xi}, y - u\rho, \tilde{z}) \tilde{\xi}^f d\tilde{\xi} du d\tilde{z}. \end{aligned}$$

Pulling out the supremum of f and estimating $h \cdot G \cdot \tilde{\xi}^f$ against a constant, we find

$$|V_e^n e^{-t\Delta} f(x, y, z)| \leq C \|f\|_\infty \leq C \cdot \|f\|_{C_e^k}.$$

This estimate again accounts for the second statement of the theorem as well.

Estimates where the diagonal meets the front face

The asymptotic behaviour of the heat kernel with compact support in the neighbourhood where td meets ff is appropriately described in the projective coordinates

$$(3.1.2) \quad \eta^2 = \frac{t}{x^2}, \quad S = \frac{x - \tilde{x}}{\sqrt{t}}, \quad U = \frac{y - \tilde{y}}{\sqrt{t}}, \quad Z = \frac{x(z - \tilde{z})}{\sqrt{t}}, \quad x, \quad y, \quad z$$

where in these coordinates η and x are the defining functions of td and ff , respectively. The edge vector fields obey the following transformation rules:

$$\beta^*(x\partial_x) = -\eta\partial_\eta + \frac{1}{\eta}\partial_S + Z\partial_Z + x\partial_x, \quad \beta^*(x\partial_y) = \frac{1}{\eta}\partial_U + x\partial_y, \quad \beta^*(\partial_z) = \frac{1}{\eta}\partial_Z + \partial_z.$$

Hence by Theorem 2.2 we infer that for any $n \in \mathbb{N}_0$,

$$\beta^*(V_e^n H)(\eta, S, U, Z, x, y, z) = x^{-m}\eta^{-m-n}G(\eta, S, U, Z, x, y, z),$$

where G is bounded in its entries (in fact, infinitely vanishing as $|(S, U, Z)| \rightarrow \infty$) and has the same compact support as β^*H .

The transformation rule of the volume form is

$$\beta^*(\tilde{x}^f d\tilde{x} \, \text{dvol}_{\partial M}(\tilde{x})) = h \cdot (x\eta)^m (1 - \eta S)^f dS \, dU \, dZ,$$

where h is a bounded distribution on \mathcal{M}_h^2 . Hence for any $f \in C_e^0(M, g)$, we have

$$\begin{aligned} V_e^n e^{-t\Delta} f(x, y, z) &= \int_M V_e^n H(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) f(\tilde{x}, \tilde{y}, \tilde{z}) \tilde{x}^f d\tilde{x} \, \text{dvol}_{\partial M}(\tilde{x}) \\ &= \int \eta^{-n} (1 - \eta S)^f h \cdot G f(x(1 - \eta S), y - x\eta U, z - \eta Z) dS \, dU \, dZ. \end{aligned}$$

Pulling out the supremum of f and estimating $x^n(1 - \eta S)^f h \cdot G$ against a constant, we find

$$|V_e^n e^{-t\Delta} f(x, y, z)| \leq C \cdot t^{-n/2} \|f\|_{C_e^0}.$$

This completes the proof of estimate (1.0.1) for $k = 0$.

For the general case, let $f \in C_e^k(M, g)$ and consider the impact of $k + n$ edge derivatives. Transforming the edge derivatives into projective coordinates near where the diagonal meets the front face, we see that the only possible singular behaviour comes from components of the form $\eta^{-1}\partial_S, \eta^{-1}\partial_U$ and $\eta^{-1}\partial_Z$. We consider $\eta^{-1}\partial_S$ (the others are similar) and estimate the integral

$$\begin{aligned} F &:= \int \eta^{-1} (\partial_S \beta^* H)(\eta, S, U, Z, x, y, z) f \cdot h(x\eta)^m (1 - \eta S)^f dS \, dU \, dZ \\ &= \int \eta^{-1} (\partial_S G)(\eta, S, U, Z, x, y, z) f \cdot h(1 - \eta S)^f dS \, dU \, dZ. \end{aligned}$$

We perform integration by parts, where the boundary terms lie away from the diagonal and hence are infinitely vanishing for $t \rightarrow 0$ by the asymptotic behaviour of the heat kernel. Omitting these irrelevant terms we obtain

$$\begin{aligned} F &= \int G(\eta, S, U, Z, x, y, z) \eta^{-1} \partial_S (f \cdot h(1 - \eta S)^f) dS \, dU \, dZ \\ &= \int G [\eta^{-1} \partial_S f(x(1 - \eta S), y - x\eta U, z - \eta Z)] h(1 - \eta S)^f dS \, dU \, dZ \\ &\quad + \int G f \cdot [\eta^{-1} \partial_S h(x(1 - \eta S), y - x\eta U, z - \eta Z) (1 - \eta S)^f] dS \, dU \, dZ. \end{aligned}$$

The second integral yields no η^{-1} -singular behaviour and is estimated as above for $(k, n) = (0, 0)$. For the first integral note

$$\eta^{-1} \partial_S f(x(1 - \eta S), y - x\eta U, z - \eta Z) = -\frac{1}{(1 - \eta S)} (\tilde{x} \partial_{\tilde{x}} f)(x(1 - \eta S), y - x\eta U, z - \eta Z).$$

Since $G(\eta, S, U, Z, x, y, z)$ is bounded in its entries, and in fact infinitely vanishing as $|(S, U, Z)| \rightarrow \infty$, we can estimate

$$|F| \leq C \|f\|_{C_e^1}.$$

Repeating the same argument for $k > 1$ edge derivatives we find that the heat operator is bounded in its action on $C_e^k(M, g)$. In view of the proof for $k = 0$, this proves the estimate (1.0.1) in full generality and it remains to check (1.0.2). Recall from §2.3 that the heat operator satisfies $e^{-t\Delta} 1 = 1$ by uniqueness of solutions to the heat equation and hence $V_e^2 e^{-t\Delta} 1 = 0$. In particular,

$$\begin{aligned} V_e^2 e^{-t\Delta} f &= \int_M V_e^2 H(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) f(\tilde{x}, \tilde{y}, \tilde{z}) \tilde{x}^f d\tilde{x} d\text{vol}_{\partial M}(\tilde{x}) \\ &= \int_M V_e^2 H(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) [f(\tilde{x}, \tilde{y}, \tilde{z}) - f(x, y, z)] \tilde{x}^f d\tilde{x} d\text{vol}_{\partial M}(\tilde{x}) =: E. \end{aligned}$$

Using the same approach as above, we obtain

$$E = \int_M \eta^{-2} G[f(x(1 - \eta S), y - x\eta U, z - \eta Z) - f(x, y, z)] h(1 - \eta S)^f dS dU dZ.$$

Estimating the difference in terms of the Hölder norm of f , we find

$$\begin{aligned} |E| &\leq \|f\|_{C_e^\alpha} \int \eta^{-2} G \cdot [d((x - x\eta S, y - x\eta U, z - \eta Z), (x, y, z))]^\alpha h(1 - \eta S)^f dS dU dZ \\ &\leq C \|f\|_{C_e^\alpha} \int \eta^{-2} h(1 - \eta S)^f G \cdot (|x\eta S|^\alpha + |x\eta U|^\alpha + |(2 - \eta S)x\eta Z|^\alpha) dS dU dZ \\ &\leq C' \|f\|_{C_e^\alpha} \int \eta^{-2+\alpha} h(1 - \eta S)^f G(\eta, S, U, Z, x, y, z) (|S|^\alpha + |U|^\alpha + |Z|^\alpha) dS dU dZ. \end{aligned}$$

Since $G(\eta, S, U, Z, x, y, z)$ is bounded in its entries, and in fact infinitely vanishing as $|(S, U, Z)| \rightarrow \infty$, we can estimate

$$|E| \leq C t^{-1+\alpha/2} \|f\|_{C_e^\alpha}.$$

We have proved Theorem 1.5 for the heat operator acting on $C_e^*(M, g)$. In order to complete the proof of Theorem 1.5 it suffices to prove that the image of $C_e^k(M, g)$ under the heat operator lies in $\mathcal{C}_e^{k+2}(M, g)$. This is a direct consequence of Proposition 2.4, which asserts z -independence of the leading coefficient in the right face expansion of H . Indeed, for any fixed $t > 0$ and any $f \in C_e^k(M, g)$, Proposition 2.4 implies $(\partial_z(V_1 \cdots V_{k+2})e^{-t\Delta} f)(0, y, z) \equiv 0$ for any $V_j \in \mathcal{V}_e, j = 1, \dots, k+2$. Hence $e^{-t\Delta}(C_e^k(M, g)) \subset \mathcal{C}_e^{k+2}(M, g)$. This completes the proof.

3.2. Estimates on spaces with a complete edge metric. In this section we prove Theorem 1.6. Note that in contrast to the incomplete edge case, the heat operator here respects the additional weighting of the edge spaces. Consequently we derive more refined estimates.

Once again the main task in the proof is to check the estimates, as the continuity of $e^{-t\Delta}f$ and its derivatives are easy to obtain by the dominated convergence theorem in each of the coordinate systems that will be discussed. The proof of the mapping properties amounts to estimating edge derivatives of the heat kernel lifted to \mathcal{M}_h^2 ; these estimates are made locally in various coordinate patches of \mathcal{M}_h^2 .

The estimates for the heat kernel supported near the diagonal away from the front face or supported in the interior of \mathcal{M}_h^2 are known (e.g., [LSU67]), so our discussion reduces to the case of the heat kernel compactly supported in an open neighbourhood of the front face. Within this neighbourhood, we consider the heat kernel being supported in each of three coordinate charts: one near each of the left and right corners of the front face, for which we use coordinates (2.2.1), and one near the intersection of td and ff , for which we use coordinates (2.2.2). Although the argument is not completely symmetric in the left and right corners due to the different roles played in the integration by tilded and untilded variables, the estimation is analogous, as we indicate below. These three charts correspond to different asymptotic regimes as the variables (t, x, \tilde{x}) approach zero.

Before we specialize to the coordinate charts, we note that the estimates of Theorem 1.6 for weighted spaces follow from the estimates for unweighted spaces. Indeed, proving the estimates for weighted spaces is equivalent to proving the unweighted estimates for the conjugated operator $X^{-w}e^{-t\Delta}X^w$, where X refers to the obvious multiplication operator. Written as an integral operator, we have

$$e^{-t\Delta}f(x, y, z) = \int_M x^{-w} H(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) \tilde{x}^w \cdot f(\tilde{x}, \tilde{y}, \tilde{z}) \tilde{x}^{-b-1} d\tilde{x} d\text{vol}_{\partial M}(\tilde{x}).$$

One can check that in the coordinate systems (2.2.1) and (2.2.2), the conjugated integral kernel $x^{-w}H\tilde{x}^w$ has the same asymptotics as H at all boundary hypersurfaces of \mathcal{M}_h^2 . Thus we may assume $w = 0$ without loss of generality.

Estimates near the lower left corner of the front face

We begin by assuming that the heat kernel is supported near the left-hand corner and work in the coordinates (2.2.1)

$$\tau = \sqrt{t}, \quad s = \frac{\tilde{x}}{x}, \quad u = \frac{y - \tilde{y}}{x}, \quad x, y, z, \tilde{z}.$$

In these coordinates τ, s , and x are the defining functions of tf , lf and ff respectively. The coordinates are valid whenever (τ, s, x) are bounded as (t, x, \tilde{x}) approach zero. The edge vector fields obey the following transformation rules:

$$\beta^*(x\partial_x) = -s\partial_s - u\partial_u + x\partial_x, \quad \beta^*(x\partial_y) = \partial_u + x\partial_y, \quad \beta^*(\partial_z) = \partial_z.$$

Hence by Theorem 2.5 we find that for any $n \in \mathbb{N}_0$, $\beta^*(V_e^n H)(\tau, s, u, x, y, z, \tilde{z})$ is bounded in x and vanishing to infinite order as $\tau \rightarrow 0$ and $s \rightarrow 0$. For the transformation rule

of the volume form we compute

$$\beta^*(\tilde{x}^{-b-1} d\tilde{x} \, \text{dvol}_{\partial M}(\tilde{x})) = h \cdot s^{-b-1} ds \, du \, d\tilde{z},$$

where h is a bounded distribution on \mathcal{M}_h^2 . Hence for any $f \in C_e^k(M, g)$, we have

$$V_e^n e^{-t\Delta} f(x, y, z) = \int h \cdot \beta^*(V_e^n H)(\tau, s, u, x, y, z, \tilde{z}) f(sx, y - xu, \tilde{z}) s^{-b-1} ds \, du \, d\tilde{z}.$$

Pulling out the supremum of f , we find

$$|V_e^n e^{-t\Delta} f(x, y, z)| \leq \|f\|_\infty \int h \cdot \beta^*(V_e^n H) s^{-b-1} ds \, du \, d\tilde{z}.$$

Since $\beta^*(V_e^n H)(\tau, s, u, x, y, z, \tilde{z})$ is vanishing to infinite order as $s \rightarrow 0$, we can estimate the integral against a constant and find

$$|V_e^n e^{-t\Delta} f(x, y, z)| \leq C \|f\|_\infty \leq C \cdot \|f\|_{C_e^k},$$

for any $k \in \mathbb{N}_0$. The second statement of Theorem 1.6 follows as well, since $C_e^\alpha(M, g) \subset C_e^0(M, g)$.

Estimates near the lower right corner of the front face

In the complete case the estimates near the right corner are simplified by the infinite order vanishing of the heat kernel at the right face. We begin by assuming that the heat kernel is supported near the right-hand corner and work in the coordinates (2.2.1)

$$\tau = \sqrt{t}, \quad \tilde{s} = \frac{x}{\tilde{x}}, \quad \tilde{u} = \frac{y - \tilde{y}}{\tilde{x}}, \quad \tilde{x}, \tilde{y}, z, \tilde{z}.$$

In these coordinates τ, \tilde{s} , and \tilde{x} are the defining functions of tf , rf and ff respectively. The coordinates are valid whenever $(\tau, \tilde{s}, \tilde{x})$ are bounded as (t, x, \tilde{x}) approach zero. The edge vector fields obey the following transformation rules:

$$\beta^*(x\partial_x) = \tilde{s}\partial_{\tilde{s}}, \quad \beta^*(x\partial_y) = \tilde{s}\partial_{\tilde{u}}, \quad \beta^*(\partial_z) = \partial_z.$$

Hence by Theorem 2.5 we find that for any $n \in \mathbb{N}_0$, $\beta^*(V_e^n H)(\tau, \tilde{s}, \tilde{u}, \tilde{x}, \tilde{y}, z, \tilde{z})$ is bounded in \tilde{x} and vanishing to infinite order as $\tau \rightarrow 0$ and $\tilde{s} \rightarrow 0$. For the transformation rule of the volume form we compute

$$\beta^*(\tilde{x}^{-b-1} d\tilde{x} \, \text{dvol}_{\partial M}(\tilde{x})) = h \cdot \tilde{s}^{-1} d\tilde{s} \, d\tilde{u} \, d\tilde{z},$$

where h is a bounded distribution on \mathcal{M}_h^2 . Hence for any $f \in C_e^k(M, g)$, we have

$$V_e^n e^{-t\Delta} f(x, y, z) = \int h \cdot \beta^*(V_e^n H)(\tau, \tilde{s}, \tilde{u}, \tilde{x}, \tilde{y}, z, \tilde{z}) f(x/\tilde{s}, y - \tilde{x}\tilde{u}, \tilde{z}) \tilde{s}^{-1} d\tilde{s} \, d\tilde{u} \, d\tilde{z}.$$

Pulling out the supremum of f , we find

$$|V_e^n e^{-t\Delta} f(x, y, z)| \leq \|f\|_\infty \int h \cdot \beta^*(V_e^n H) \tilde{s}^{-1} d\tilde{s} \, d\tilde{u} \, d\tilde{z}.$$

Since $\beta^*(V_e^n H)(\tau, \tilde{s}, \tilde{u}, \tilde{x}, \tilde{y}, z, \tilde{z})$ is vanishing to infinite order as $\tilde{s} \rightarrow 0$, we can estimate the integral against a constant and find

$$|V_e^n e^{-t\Delta} f(x, y, z)| \leq C \|f\|_\infty \leq C \cdot \|f\|_{C_e^k},$$

for any $k \in \mathbb{N}_0$. The second statement of Theorem 1.6 follows as well, since $C_e^\alpha(M, g) \subset C_e^0(M, g)$.

Estimates near the diagonal

The asymptotic behaviour of the heat kernel with compact support in the neighbourhood where the diagonal meets the front face is appropriately described in the projective coordinates (2.2.2)

$$\eta = \sqrt{t}, \quad S = \frac{x - \tilde{x}}{x\sqrt{t}}, \quad U = \frac{y - \tilde{y}}{x\sqrt{t}}, \quad Z = \frac{z - \tilde{z}}{\sqrt{t}}, \quad x, y, z.$$

Recall that in these coordinates η and x are the defining functions of td and ff , respectively. The edge vector fields obey the following transformation rules:

$$\beta^*(x\partial_x) = (1 - \eta S)\frac{1}{\eta}\partial_S - U\partial_U + x\partial_x, \quad \beta^*(x\partial_y) = \frac{1}{\eta}\partial_U + x\partial_y, \quad \beta^*(\partial_z) = \frac{1}{\eta}\partial_Z + \partial_z.$$

Hence by Theorem 2.5 we infer that for any $n \in \mathbb{N}_0$,

$$\beta^*(V_e^n H)(\eta, S, U, Z, x, y, z) = \eta^{-m-n} G(\eta, S, U, Z, x, y, z),$$

where G is bounded in its entries, and in fact infinitely vanishing as $|(S, U, Z)| \rightarrow \infty$, with the same compact support as β^*H . The coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ on the second copy of M are expressed in terms of the projective coordinates (2.2.2) by

$$\tilde{x} = x(1 - \eta S), \quad \tilde{y} = y - x\eta U, \quad \tilde{z} = z - \eta Z.$$

Thus the transformation rule of the volume form is

$$\beta^*(\tilde{x}^{-b-1} d\tilde{x} \, \text{dvol}_{\partial M}(\tilde{x})) = h\eta^m (1 - \eta S)^{-b-1} dS \, dU \, dZ,$$

where h is a bounded distribution on \mathcal{M}_h^2 . Hence for any $f \in C_e^0(M, g)$, we have

$$\begin{aligned} D &:= \int_M V_e^n H(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) f(\tilde{x}, \tilde{y}, \tilde{z}) \tilde{x}^{-b-1} d\tilde{x} \, \text{dvol}_{\partial M}(\tilde{x}) \\ &= \int \eta^{-n} (1 - \eta S)^{-b-1} h \cdot G f(x(1 - \eta S), y - x\eta U, z - \eta Z) dS \, dU \, dZ. \end{aligned}$$

Pulling out the supremum of f and estimating $(1 - \eta S)^{-b-1} h \cdot G$ against a constant, we find

$$D \leq C \cdot t^{-n/2} \|f\|_{C_e^0}.$$

This completes the proof of estimate (1.0.3) for $k = 0$. For the general case, let $f \in C_e^k(M, g)$ and consider the impact of $(k + n)$ edge derivatives. Transforming the edge derivatives into projective coordinates near the intersection of the diagonal and the front face, we see that the only possible singular behaviour comes from components of the form $\eta^{-1}\partial_S, \eta^{-1}\partial_U$ and $\eta^{-1}\partial_Z$. We consider $\eta^{-1}\partial_S$ (the others are similar) and estimate the integral

$$\begin{aligned} F &:= \int \eta^{-1} (\partial_S \beta^* H)(\eta, S, U, Z, x, y, z) f \cdot h\eta^m (1 - \eta S)^{-b-1} dS \, dU \, dZ \\ &= \int \eta^{-1} (\partial_S G)(\eta, S, U, Z, x, y, z) f \cdot h(1 - \eta S)^{-b-1} dS \, dU \, dZ. \end{aligned}$$

We integrate by parts and note that G is infinitely vanishing for $S \rightarrow \pm\infty$ by the asymptotic behaviour of the heat kernel. Omitting this boundary term we obtain

$$\begin{aligned} F &= \int G(\eta, S, U, Z, x, y, z) \eta^{-1} \partial_S (f \cdot h(1 - \eta S)^{-b-1}) dS dU dZ \\ &= \int G [\eta^{-1} \partial_S f(x(1 - \eta S), y - x\eta U, z - \eta Z)] h(1 - \eta S)^{-b-1} dS dU dZ \\ &\quad + \int G f \cdot [\eta^{-1} \partial_S h(x(1 - \eta S), y - x\eta U, z - \eta Z) (1 - \eta S)^{-b-1}] dS dU dZ. \end{aligned}$$

After differentiating in the second integral we see that no η^{-1} term remains, and this integral can be estimated using the same approach as for D above. For the first integral note that

$$\eta^{-1} \partial_S f(x(1 - \eta S), y - x\eta U, z - \eta Z) = -\frac{1}{(1 - \eta S)} (\tilde{x} \partial_{\tilde{x}} f)(x(1 - \eta S), y - x\eta U, z - \eta Z).$$

The edge derivative $\tilde{x} \partial_{\tilde{x}} f$ is bounded above by $\|f\|_{C_e^1}$; the remaining terms in the integral can be estimated as for D since $G(\eta, S, U, Z, x, y, z)$ is bounded in its entries and in fact infinitely vanishing as $|(S, U, Z)| \rightarrow \infty$. Thus $|F| \leq C\|f\|_{C_e^1}$.

Repeating the same argument for $k > 1$ edge derivatives we find that the heat operator is bounded in its action on $C_e^k(M, g)$. In view of the proof of estimate (1.0.3) for $k = 0$, the first statement of Theorem 1.6 holds in full generality and it remains to check the second statement. Recall from Section 2.3 that the heat operator satisfies $e^{-t\Delta} 1 = 1$ by stochastic completeness, and hence $V_e^2 e^{-t\Delta} 1 = 0$. In particular,

$$\begin{aligned} V_e^2 e^{-t\Delta} f &= \int_M V_e^2 H(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) f(\tilde{x}, \tilde{y}, \tilde{z}) \tilde{x}^{-b-1} d\tilde{x} d\text{vol}_{\partial M}(\tilde{x}) \\ &= \int_M V_e^2 H(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) [f(\tilde{x}, \tilde{y}, \tilde{z}) - f(x, y, z)] \tilde{x}^{-b-1} d\tilde{x} d\text{vol}_{\partial M}(\tilde{x}) =: E. \end{aligned}$$

Using the same approach as for D above, we obtain

$$E = \int \eta^{-2} G [f(x(1 - \eta S), y - x\eta U, z - \eta Z) - f(x, y, z)] h(1 - \eta S)^{-b-1} dS dU dZ.$$

Estimating the difference in terms of the Hölder norm of f , we find

$$\begin{aligned} |E| &\leq \|f\|_{C_e^\alpha} \int \eta^{-2} G \cdot d((x - x\eta S, y - x\eta U, z - \eta Z), (x, y, z))^\alpha h(1 - \eta S)^{-b-1} dS dU dZ \\ &\leq C\|f\|_{C_e^\alpha} \int \eta^{-2} h(1 - \eta S)^{-b-1} G \cdot \left(\left| \frac{\eta S}{(2 - \eta S)} \right|^\alpha + \left| \frac{\eta U}{(2 - \eta S)} \right|^\alpha + |\eta Z|^\alpha \right) dS dU dZ \\ &\leq C'\|f\|_{C_e^\alpha} \int \eta^{-2+\alpha} G(\eta, S, U, Z, x, y, z) (|S|^\alpha + |U|^\alpha + |Z|^\alpha) dS dU dZ. \end{aligned}$$

Since $G(\eta, S, U, Z, x, y, z)$ is bounded in its entries, and in fact infinitely vanishing as $|(S, U, Z)| \rightarrow \infty$, we can estimate

$$|E| \leq C t^{-1+\alpha/2} \|f\|_{C_e^\alpha}.$$

The proof of Theorem 1.6 is now complete.

4. SHORT-TIME EXISTENCE OF SOLUTIONS TO SEMILINEAR PARABOLIC EQUATIONS

In this section we provide an application of our estimates to prove short-time existence of solutions to some semilinear parabolic equations on both complete and incomplete edge spaces. We follow [JL03, JL04] closely. The underlying idea is based on [Tay96].

Consider an equation of the form

$$(4.0.1) \quad \begin{cases} \partial_t u - \Delta u &= Q(u, V_e u), \\ u(0) &= u_0, \end{cases}$$

where Δ denotes the appropriate edge Laplacian and Q denotes inhomogeneous terms that involve up to one edge derivative of u . Applying Duhamel's principle, we transform the problem to an equivalent integral equation:

$$u = e^{-t\Delta} u_0 + \int_0^t e^{-(t-s)\Delta} Q(u, V_e u) ds.$$

With appropriate assumptions and the estimates of Theorems 1.5 or 1.6, this equation may be solved by the contraction mapping principle, as we now explain.

Recall that a semigroup S_t of operators on a Banach space X is strongly continuous if for all $f \in X$,

$$\|S_t f - f\|_X \longrightarrow 0 \text{ as } t \longrightarrow 0.$$

Taylor proves:

Theorem 4.1. [Tay96, Proposition 15.1.1] *Suppose that X and Y are Banach spaces such that:*

- (i) $e^{-t\Delta} : X \longrightarrow X$ is a strongly continuous semigroup, for $t \geq 0$.
- (ii) $Q : X \longrightarrow Y$ is locally Lipschitz.
- (iii) $e^{-t\Delta} : Y \longrightarrow X$, for $t > 0$.
- (iv) For some $\gamma < 1$,

$$\|e^{-t\Delta}\|_{L(Y,X)} \leq Ct^{-\gamma}.$$

Then for any $u_0 \in Y$, the initial value problem (4.0.1) has a unique solution $u \in C([0, T], X)$, for some $T > 0$. T may be estimated from below in terms of $\|f\|_X$.

In view of this theorem and our previous estimates, it remains to prove strong continuity of the semigroup $e^{-t\Delta}$ on either $X = \mathcal{C}_e^k(M, g)$ or $X = C_e^k(M, g)$, as appropriate, to deduce short-time existence of solutions to (4.0.1) for certain locally Lipschitz Q . See [GY09] for the classical proof.

Proposition 4.2. *Let (M, g) be an m -dimensional Riemannian manifold with a feasible incomplete non-iterated edge metric. Let $e^{-t\Delta}$ denote the heat operator corresponding to the Friedrichs extension Δ of the associated Laplacian on (M, g) . Then $e^{-t\Delta}$ is strongly continuous on $\mathcal{C}_e^k(M, g)$, for any $k \geq 0$.*

Proof. We prove the statement by adapting the classical proof of strong continuity of the heat operator on closed (non-singular) manifolds to the present setup. Assume first $k = 0$. Using stochastic completeness of the heat kernel we find

$$(e^{-t\Delta} f)(p, t) - f(p) = \int_M H(t, p, \tilde{p}) (f(\tilde{p}) - f(p)) \, \text{dvol}_g(\tilde{p}).$$

Note that $f \in \mathcal{C}_e^0(M, g)$ and hence for any $\epsilon > 0$ there exists some $\delta(\epsilon) > 0$, such that for $d(p, \tilde{p}) \leq \delta(\epsilon)$ one has $|f(p) - f(\tilde{p})| \leq \epsilon$. For any given $\epsilon > 0$ we separate the integration region into

$$(4.0.2) \quad \begin{aligned} M_\epsilon^+ &:= \{\tilde{p} \mid d(p, \tilde{p}) \geq \delta(\epsilon)\}, \\ M_\epsilon^- &:= \{\tilde{p} \mid d(p, \tilde{p}) \leq \delta(\epsilon)\}. \end{aligned}$$

Employing continuity of f we find

$$\begin{aligned} |e^{-t\Delta}f - f| &= \left| \int_M H(t, p, \tilde{p}) (f(\tilde{p}) - f(p)) \, d\text{vol}_g(\tilde{p}) \right| \\ &\leq \int_{M^+} H(t, p, \tilde{p}) |f(\tilde{p}) - f(p)| \, d\text{vol}_g(\tilde{p}) + \int_{M^-} H(t, p, \tilde{p}) |f(\tilde{p}) - f(p)| \, d\text{vol}_g(\tilde{p}) \\ &\leq 2 \frac{\sqrt{t}}{\delta(\epsilon)} \|f\|_0 \int_{M^+} H(t, p, \tilde{p}) \frac{d(p, \tilde{p})}{\sqrt{t}} \, d\text{vol}_g(\tilde{p}) + \epsilon \int_{M^-} H(t, p, \tilde{p}) \, d\text{vol}_g(\tilde{p}). \end{aligned}$$

The second integral is bounded by ϵ . We must verify in projective coordinates that the first integral is bounded, uniformly in (t, p, ϵ) . It suffices to assume the heat kernel is supported near the front face, since away from ff the estimates reduce to the classical case of a manifold without boundary near the diagonal. Assume first that the heat kernel is supported in a neighbourhood where td meets ff, and use the projective coordinates (3.1.2):

$$\eta^2 = \frac{t}{x^2}, \quad S = \frac{x - \tilde{x}}{\sqrt{t}}, \quad U = \frac{y - \tilde{y}}{\sqrt{t}}, \quad Z = \frac{x(z - \tilde{z})}{\sqrt{t}}, \quad x, y, z.$$

In these coordinates η and x are the defining functions of td and ff, respectively. By Theorem 2.2 we infer

$$\beta^* H(\eta, S, U, Z, x, y, z) = x^{-m} \eta^{-m} G(\eta, S, U, Z, x, y, z),$$

where G is bounded in its entries, and in fact infinitely vanishing as $|(S, U, Z)| \rightarrow \infty$, with the same compact support as $\beta^* H$. The transformation rule for the volume form is

$$\beta^*(\tilde{x}^f d\tilde{x} \, d\text{vol}_{\partial M}(\tilde{x})) = h(x\eta)^m (1 - \eta S)^f dS \, dU \, dZ,$$

where $h = h(\eta, x(1 - \eta S), y - x\eta U, z - \eta Z, x, y, z)$ is a bounded distribution on \mathcal{M}_h^2 . We may estimate $d(p, \tilde{p})/\sqrt{t}$:

$$\frac{d(p, \tilde{p})}{\sqrt{t}} \leq C \sqrt{|S|^2 + |U|^2 + |Z|^2}.$$

Hence we find

$$\int_{M^+} H(t, p, \tilde{p}) \frac{d(p, \tilde{p})}{\sqrt{t}} \, d\text{vol}_g(\tilde{p}) \leq C' \int h(1 - \eta S)^f G \cdot \sqrt{|S|^2 + |U|^2 + |Z|^2} \, dS \, dU \, dZ \leq C'',$$

where C'' is independent of (t, x, y, z) and ϵ . Entirely analogous estimation works near the top corner of the front face. In the estimate near the left corner (resp. right corner) of the front face one finds that $d(p, \tilde{p})/\sqrt{t}$ is bounded by a power of τ^{-1} (resp. $\tilde{\tau}^{-1}$), which may be absorbed into the heat kernel as it vanishes to infinite order as $\tau \rightarrow 0$ (resp. $\tilde{\tau} \rightarrow 0$) at this corner. Therefore we obtain

$$\|e^{-t\Delta}f - f\|_0 \leq C'' \frac{\sqrt{t}}{\delta(\epsilon)} \|f\|_0 + \epsilon.$$

Thus, for any given $\epsilon > 0$ we can estimate $\|e^{-t\Delta}f - f\|_0 \leq 2\epsilon$ for $\sqrt{t} < \epsilon\delta(\epsilon)/(C''\|f\|_0)$. This proves strong continuity of the heat operator on $\mathcal{C}_e^0(M, g)$.

We prove strong continuity on $\mathcal{C}_e^k(M, g)$ for $k \geq 1$ in a similar fashion. Near the diagonal it requires an integration by parts argument like the one in §3.1. The estimates away from the diagonal are the same as for $\mathcal{C}_e^0(M, g)$, so we assume that β^*H is compactly supported in a neighbourhood where td meets ff . The edge vector fields obey the following transformation rules:

$$\beta^*(x\partial_x) = -\eta\partial_\eta + \frac{1}{\eta}\partial_S + Z\partial_Z + x\partial_x, \quad \beta^*(x\partial_y) = \frac{1}{\eta}\partial_U + x\partial_y, \quad \beta^*(\partial_z) = \frac{1}{\eta}\partial_Z + \partial_z.$$

We consider $\|x\partial_x(e^{-t\Delta}f - f)\|_0$. Using stochastic completeness of the heat kernel, we find

$$\begin{aligned} F &:= x\partial_x(e^{-t\Delta}f - f) = \int (x\partial_x H)f(\tilde{x}, \tilde{y}, \tilde{z})\tilde{x}^f d\tilde{x} d\text{vol}_{\partial M}(\tilde{x}) \\ &\quad - \int (x\partial_x)[Hf(x, y, z)\tilde{x}^f d\tilde{x} d\text{vol}_{\partial M}(\tilde{x})] =: F_1 - F_2. \end{aligned}$$

Next we transform to projective coordinates and integrate by parts in S , where the boundary terms lie away from the diagonal and hence are infinitely vanishing for $t \rightarrow 0$ by the asymptotic behaviour of the heat kernel. Omitting these irrelevant terms, we obtain

$$\begin{aligned} F_1 &= \int \left(-\eta\partial_\eta + \frac{1}{\eta}\partial_S + Z\partial_Z + x\partial_x \right) [(x\eta)^{-m}G(\eta, S, U, Z, x, y, z)] \\ &\quad \times f(x(1-\eta S), y - x\eta U, z - \eta Z) h(x\eta)^m (1-\eta S)^f dS dU dZ \\ &= \int [(-\eta\partial_\eta + Z\partial_Z + x\partial_x)(x\eta)^{-m}G] \cdot f h(x\eta)^m (1-\eta S)^f dS dU dZ \\ &\quad - \int G \left[\left(\frac{1}{\eta}\partial_S \right) f \right] h(1-\eta S)^f dS dU dZ \\ &\quad - \int (x\eta)^{-m}G \cdot f \left[\left(\frac{1}{\eta}\partial_S \right) h(x\eta)^m (1-\eta S)^f \right] dS dU dZ. \end{aligned}$$

We perform similar computations for F_2 :

$$\begin{aligned}
F_2 &= \int \left[(x\partial_x H)f(x, y, z) + H(x\partial_x f) \right] \tilde{x}^f d\tilde{x} d\text{vol}_{\partial M}(\tilde{x}) \\
&= \int \left(\left[-\eta\partial_\eta + \frac{1}{\eta}\partial_S + Z\partial_Z + x\partial_x \right] (x\eta)^{-m} G \right) f \cdot h(x\eta)^m (1 - \eta S)^f dS dU dZ \\
&\quad + \int G(\eta, S, U, Z, x, y, z) (x\partial_x f(x, y, z)) h(1 - \eta S)^f dS dU dZ \\
&= \int \left[(-\eta\partial_\eta + Z\partial_Z + x\partial_x) (x\eta)^{-m} G \right] \cdot f h(x\eta)^m (1 - \eta S)^f dS dU dZ \\
&\quad - \int (x\eta)^{-m} G \cdot f \left[\left(\frac{1}{\eta}\partial_S \right) h(x\eta)^m (1 - \eta S)^f \right] dS dU dZ \\
&\quad + \int G(\eta, S, U, Z, x, y, z) (x\partial_x f(x, y, z)) h(1 - \eta S)^f dS dU dZ.
\end{aligned}$$

Thus $F = F_1 - F_2$ becomes

$$\begin{aligned}
F &= \int \left[(-\eta\partial_\eta + Z\partial_Z + x\partial_x) (x\eta)^{-m} G(\eta, S, U, Z, x, y, z) \right] h(x\eta)^m (1 - \eta S)^f \\
&\quad \times (f(x(1 - \eta S), y - x\eta U, z - \eta Z) - f(x, y, z)) dS dU dZ \\
&\quad - \int G(\eta, S, U, Z, x, y, z) \left[\left(\frac{1}{\eta}\partial_S \right) h \cdot (1 - \eta S)^f \right] \\
&\quad \times (f(x(1 - \eta S), y - x\eta U, z - \eta Z) - f(x, y, z)) dS dU dZ \\
&\quad - \int G \left[\frac{1}{\eta}\partial_S f(x(1 - \eta S), y - x\eta U, z - \eta Z) + x\partial_x f(x, y, z) \right] h(1 - \eta S)^f dS dU dZ.
\end{aligned}$$

Now, each of the three integrals is estimated as above for $k = 0$ by separating the integration region into M_ϵ^+ and M_ϵ^- for any given $\epsilon > 0$. Note that in the final integral we use the fact that $f \in \mathcal{C}_e^1(M, g)$. Higher order and other edge derivatives may be estimated in a similar way. \square

Proposition 4.3. *Let (M, g) be an m -dimensional Riemannian manifold with a feasible complete edge metric. Let $e^{-t\Delta}$ denote the heat operator of the unique self-adjoint extension Δ of the associated Laplacian on (M, g) . Then $e^{-t\Delta}$ is strongly continuous on $C_e^k(M, g)$, for any $k \geq 0$.*

Proof. We proceed as in the incomplete edge case, again modeling the proof after the classical estimate on a closed manifold. We estimate near td and ff ; the estimation near either the left corner or right corner is simpler than what follows due to the infinite order vanishing of the heat kernel at tf .

We first consider the case $k = 0$ and show $\|e^{-t\Delta}f - f\|_0 \rightarrow 0$ as $t \rightarrow 0$. Using stochastic completeness of the heat kernel we find

$$(e^{-t\Delta}f)(p, t) - f(p) = \int_M H(t, p, \tilde{p}) (f(\tilde{p}) - f(p)) d\text{vol}_g(\tilde{p}).$$

Recall that $f \in C_e^0(M)$ means f is continuous on a compact manifold with boundary, so for every $\epsilon > 0$ we obtain a uniform $\delta = \delta(\epsilon)$, where $d(p, \tilde{p}) < \delta$ implies $|f(p) - f(\tilde{p})| < \epsilon$. We separate the integration region into M_ϵ^+ and M_ϵ^- as in (4.0.2) and find

$$\begin{aligned} |e^{-t\Delta} f - f| &= \left| \int_M H(t, p, \tilde{p}) (f(\tilde{p}) - f(p)) \, d\text{vol}_g(\tilde{p}) \right| \\ &\leq \int_{M^+} H(t, p, \tilde{p}) |f(\tilde{p}) - f(p)| \, d\text{vol}_g(\tilde{p}) + \int_{M^-} H(t, p, \tilde{p}) |f(\tilde{p}) - f(p)| \, d\text{vol}_g(\tilde{p}) \\ &\leq 2 \frac{\sqrt{t}}{\delta(\epsilon)} \|f\|_0 \int_{M^+} H(t, p, \tilde{p}) \frac{d(p, \tilde{p})}{\sqrt{t}} \, d\text{vol}_g(\tilde{p}) + \epsilon. \end{aligned}$$

Assume that the heat kernel is supported in a neighbourhood where td meets ff , and use the appropriate projective coordinates (2.2.2)

$$\eta = \sqrt{t}, \quad S = \frac{x - \tilde{x}}{x\sqrt{t}}, \quad U = \frac{y - \tilde{y}}{x\sqrt{t}}, \quad Z = \frac{z - \tilde{z}}{\sqrt{t}}, \quad x, y, z.$$

In these coordinates η and x are the defining functions of td and ff , respectively. The edge vector fields obey the transformation rules:

$$\beta^*(x\partial_x) = (1 - \eta S) \frac{1}{\eta} \partial_S - U \partial_U + x \partial_x, \quad \beta^*(x\partial_y) = \frac{1}{\eta} \partial_U + x \partial_y, \quad \beta^*(\partial_z) = \frac{1}{\eta} \partial_Z + \partial_z.$$

Hence, by Theorem 2.5 we infer

$$\beta^* H(\eta, S, U, Z, x, y, z) = \eta^{-m} G(\eta, S, U, Z, x, y, z),$$

where G is bounded in its entries, and in fact infinitely vanishing as $|(S, U, Z)| \rightarrow \infty$, with the same compact support as $\beta^* H$. The coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ on the second copy of M are expressed in terms of the new projective coordinates by

$$\tilde{x} = x(1 - \eta S), \quad \tilde{y} = y - x\eta U, \quad \tilde{z} = z - \eta Z.$$

From there we compute the transformation rule of the volume form

$$\beta^*(\tilde{x}^{-b-1} d\tilde{x} \, d\text{vol}_{\partial M}(\tilde{x})) = h \eta^m (1 - \eta S)^{-b-1} dS \, dU \, dZ,$$

where $h = h(\eta, x(1 - \eta S), y - x\eta U, z - \eta Z, x, y, z)$ is a bounded distribution on \mathcal{M}_h^2 whose arguments we will suppress. Further, we can estimate

$$\frac{d(p, \tilde{p})}{\sqrt{t}} \leq \sqrt{\frac{S^2}{(2 - \eta S)^2} + \frac{U^2}{(2 - \eta S)^2} + Z^2} \leq C \sqrt{S^2 + U^2 + Z^2}.$$

Hence we can write after cancellations

$$\begin{aligned} |e^{-t\Delta} f - f| &\leq 2 \frac{\sqrt{t}}{\delta(\epsilon)} \|f\|_0 \int_{M^+} H(t, p, \tilde{p}) \frac{d(p, \tilde{p})}{\sqrt{t}} \, d\text{vol}_g(\tilde{p}) + \epsilon \\ &\leq C \frac{\sqrt{t}}{\delta(\epsilon)} \|f\|_0 \int h(1 - \eta S)^{-b-1} \cdot G \cdot \sqrt{S^2 + U^2 + Z^2} dS \, dU \, dZ + \epsilon. \\ &\leq C \frac{\sqrt{t}}{\delta(\epsilon)} \|f\|_0 + \epsilon, \end{aligned}$$

where C is independent of (p, t) (see §3.2 for similar estimation near where td meets ff). As in Proposition 4.2, we conclude

$$\lim_{t \rightarrow 0} \|e^{-t\Delta} f - f\|_0 = 0,$$

completing the case $k = 0$.

To prove strong continuity on $C_e^k(M, g)$ for $k \geq 1$ requires an integration by parts argument similar to the one in Proposition 4.2. We consider $\|x\partial_x(e^{-t\Delta} f - f)\|_0$. Using stochastic completeness of the heat kernel, we find

$$\begin{aligned} F &:= x\partial_x(e^{-t\Delta} f - f) = \int (x\partial_x H) \cdot f(\tilde{x}, \tilde{y}, \tilde{z}) \tilde{x}^{-b-1} d\tilde{x} d\text{vol}_{\partial M}(\tilde{x}) \\ &\quad - \int (x\partial_x)[H(t, x, y, z, \tilde{x}, \tilde{y}, \tilde{z})f(x, y, z)] \tilde{x}^{-b-1} d\tilde{x} d\text{vol}_{\partial M}(\tilde{x}) =: F_1 - F_2. \end{aligned}$$

Next we transform to projective coordinates and integrate by parts in S , where the boundary terms lie away from the diagonal and hence are infinitely vanishing for $t \rightarrow 0$ by the asymptotic behaviour of the heat kernel. Omitting these irrelevant terms, we obtain

$$\begin{aligned} F_1 &= \int \left((1 - \eta S) \frac{1}{\eta} \partial_S - U \partial_U + x \partial_x \right) [\eta^{-m} G(\eta, S, U, Z, x, y, z)] \\ &\quad \times f(x(1 - \eta S), y - x\eta U, z - \eta Z) h \eta^m (1 - \eta S)^{-b-1} dS dU dZ \\ &= \int [(-U \partial_U + x \partial_x) \eta^{-m} G] \cdot f h \eta^m (1 - \eta S)^{-b-1} dS dU dZ \\ &\quad - \int G \cdot (\partial_S f) \cdot h \eta^{-1} (1 - \eta S)^{-b} dS dU dZ \\ &\quad - \int G \cdot f \cdot \partial_S \eta^{-1} (h(1 - \eta S)^{-b}) dS dU dZ. \end{aligned}$$

We perform similar computations for F_2 :

$$\begin{aligned} F_2 &= \int \left((x\partial_x H) f(x, y, z) + H \cdot (x\partial_x f) \right) \tilde{x}^{-b-1} d\tilde{x} d\text{vol}_{\partial M}(\tilde{x}) \\ &= \int \left((1 - \eta S) \frac{1}{\eta} \partial_S - U \partial_U + x \partial_x \right) [\eta^{-m} G] f \cdot h \eta^m (1 - \eta S)^{-b-1} dS dU dZ \\ &\quad + \int G(\eta, S, U, Z, x, y, z) (x\partial_x f(x, y, z)) h (1 - \eta S)^{-b-1} dS dU dZ \\ &= \int [(-U \partial_U + x \partial_x) \eta^{-m} G] \cdot f(x, y, z) h \eta^m (1 - \eta S)^{-b-1} dS dU dZ \\ &\quad - \int G \cdot f \cdot \partial_S (\eta^{-1} h (1 - \eta S)^{-b}) dS dU dZ \\ &\quad + \int G(\eta, S, U, Z, x, y, z) (x\partial_x f(x, y, z)) h (1 - \eta S)^{-b-1} dS dU dZ. \end{aligned}$$

Carefully recalling the variables we have suppressed, we see that $F = F_1 - F_2$ becomes

$$\begin{aligned} F &= \int \left[(-U\partial_U + x\partial_x)\eta^{-m}G(\eta, S, U, Z, x, y, z) \right] h\eta^m(1 - \eta S)^{-b-1} \\ &\quad \times (f(x(1 - \eta S), y - x\eta U, z - \eta Z) - f(x, y, z))dS dU dZ \\ &\quad - \int G(\eta, S, U, Z, x, y, z)(\partial_S(\eta^{-1}h(x(1 - \eta S), y - x\eta U, z - \eta Z)(1 - \eta S)^{-b})) \\ &\quad \times (f(x(1 - \eta S), y - x\eta U, z - \eta Z) - f(x, y, z))dS dU dZ \\ &\quad - \int G \cdot ((1 - \eta S)\partial_S(\eta^{-1}f(x(1 - \eta S), y - x\eta U, z - \eta Z)) + x\partial_x f(x, y, z)) \cdot \\ &\quad \times h(1 - \eta S)^{-b-1}dS dU dZ. \end{aligned}$$

All three integrals are now estimated as in the $k = 0$ case. Note that in the second integral, the apparently singular η^{-1} is cancelled upon first performing the differentiation; in the third integral, we use the fact that $f \in C_e^1(M, g)$. Higher order and other edge derivatives may be similarly estimated. \square

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